COMPLEX ANALYSIS
TOPIC III: COMPLEX NUMBERS

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1. Complex Algebra

The set of *complex numbers* is

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}, i^2 = -1\}.$$

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Define addition and multiplication in $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i;$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

Thus to add or multiply complex numbers, treat i like a variable, add or multiply, replace i^2 with -1, and combine like terms.

We know a number is complex when it is in *standard form* $z = x + yi$. That is, we should be able to identify the real numbers x and y. Always put a complex number in standard form to complete a computation.

Example 1. Let $z = 2 + 5i$ and $w = 3 - 7i$. Then

$$z + w = (2 + 5i) + (3 - 7i) = (2 + 3) + (5i - 7i) = 5 - 2i,$$

and

$$zw = (2 + 5i)(3 - 7i) = 6 - 14i + 15i - 35i^2 = 6 + i + 35 = 41 + i.$$

It is also possible to divide complex numbers, as follows.

If $z = x + yi$, then *conjugate* of z is denoted $\overline{z}$, and is given by $\overline{z} = x - yi$. This is useful, among other reasons, for complex division. To simplify a fraction of complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

Example 2. Let $z = 2 + 5i$ and $w = 3 - 7i$. Then

$$\frac{z}{w} = \frac{2 + 5i}{3 - 7i} = \frac{(2 + 5i)(3 + 7i)}{(3 - 7i)(3 + 7i)} = \frac{6 + 14i + 15i + 35i^2}{9 + 21i - 21i - 49i^2} = \frac{-29 + 29i}{58} = -\frac{1}{2} + \frac{1}{2}i.$$

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2. Complex Geometry

2.1. **Complex Numbers as Vectors.** Let $z = x + iy$ be an arbitrary complex number. The *real part* of z is $\mathrm{Re}(z) = x$. The *imaginary part* of z is $\mathrm{Im}(z) = y$. We view $\mathbb{R}$ as the subset of $\mathbb{C}$ consisting of those elements whose imaginary part is zero.

We graph complex number on the xy-plane, using the real part as the first coordinate and the imaginary part as the second coordinate. This sets up a bijective (one-to-one and onto) correspondence $\mathbb{C} \leftrightarrow \mathbb{R}^2$. So just as the real numbers are viewed geometrically as a line, the complex numbers are viewed geometrically as a plane, typically referred to as the *complex plane*.

Under this interpretation, the set $\mathbb{C}$ may be identified with the set of all vectors in $\mathbb{R}^2$. That is, if $z = x + yi \in \mathbb{C}$, z corresponds to the vector $\langle x, y\rangle$.

2.2. **Geometric Interpretation of Complex Addition.** Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$. Then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i;$$

since the corresponding vector sum is

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1, x_2, y_1, y_2\rangle,$$

we see that complex addition corresponds to vector addition. Thus the geometric interpretation of complex addition is vector addition.

Similarly, let $t \in \mathbb{R}$ and $z = x + yi$. Then $t = t + 0i$ is a complex number, and

$$tz = (t + 0i)(x + yi) = tx + tyi;$$

the corresponding scalar multiplication is

$$t\langle x, y\rangle = \langle tx, ty\rangle.$$

Thus the geometric interpretation of multiplying a real number times a complex number is scalar multiplication.

3. Complex Conjugation

Seeing complex numbers are vectors immediately gives us the notions of the length and angle of a complex number.

The *modulus* of z is

$$|z| = \sqrt{x^2 + y^2}.$$

This is the length of z as a vector.

The *angle* of z, denoted by $\angle(z)$, is the angle between the vectors $(1, 0)$ and (x, y) in the real plane $\mathbb{R}^2$; this is well-defined up to a multiple of 2π.

Let $z = x + iy$ be an arbitrary complex number. The *conjugate* of z is

$$\overline{z} = x - iy.$$

This is the mirror image of z under reflection across the real axis. There are a number of identities which involve complex conjugation.

(a) $z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}(z)$
(b) $z - \overline{z} = ((x + iy) - (x - iy) = 2yi = 2\,\mathrm{Im}(z)i$
(c) $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$
(d) $\dfrac{z}{\overline{z}} = \dfrac{z^2}{z\overline{z}} = \dfrac{z^2}{|z|^2} = \left(\dfrac{z}{|z|}\right)^2$

3.1. **Geometric Interpretation of Complex Multiplication.** Let $r = |z|$ and $\theta = \angle(z)$. Then $x = r\cos\theta$ and $y = r\sin\theta$. Define a function

$$\operatorname{cis} : \mathbb{R} \to \mathbb{C} \quad \text{by} \quad \operatorname{cis}(\theta) = \cos\theta + i\sin\theta.$$

Then $z = r\operatorname{cis}(\theta)$; this is the *polar representation* of z. Note that $\frac{z}{|z|} = \operatorname{cis}\theta$.

Recall the trigonometric formulae for the cosine and sine of the sum of angles:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad \text{and} \quad \sin(A+B) = \cos A\sin B + \sin A\cos B.$$

Let $z_1 = r_1\operatorname{cis}(\theta_1)$ and $z_2 = r_2\operatorname{cis}(\theta_2)$. Then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ &= r_1r_2\operatorname{cis}(\theta_1+\theta_2). \end{aligned}$$

Thus the geometric interpretation of complex multiplication is:

- **(a)** The radius of the product is the product of the radii;
- **(b)** The angle of the product is the sum of the angles.

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 2z$. Then f dilates the complex plane by a factor of 2.

Example 4. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz$. Then f rotates the complex plane by 90 degrees.

Example 5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = (1+i)z$. Note that $|1+i| = \sqrt{2}$ and $\angle(1+i) = \frac{\pi}{4}$. Then f dilates the complex plane by a factors of $\sqrt{2}$ and rotates it by 45 degrees.

Example 6. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = az$, where a is a fixed complex number. Then f dilates the plane by a factor of $|a|$ and rotates the plane by an angle of $\angle(a)$.

4. Complex Powers and Roots

A special case of complex multiplication is exponentiation by a natural number; a simple proof by induction shows that

Theorem 1. (DeMoivre's Theorem)
Let $\theta \in \mathbb{R}$. Then

$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

Let $z = r\operatorname{cis}(\theta)$ and let $n \in \mathbb{N}$. Then $z^n = r^n\operatorname{cis}(n\theta)$.

The *unit circle* in the complex plane is

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Note that if $u_1, u_2 \in \mathbb{U}$, then $u_1u_2 \in \mathbb{U}$.

Let $\zeta \in \mathbb{C}$ and suppose that $\zeta^n = 1$. We call ζ an n^{th} *root of unity.* If $\zeta^m \neq 1$ for $m \in \{1, \dots, n-1\}$, we call ζ a *primitive* n^{th} *root of unity.*

Let $\zeta = \operatorname{cis}(\frac{2\pi}{n})$. Then $\zeta^n = \operatorname{cis}(n\frac{2\pi}{n}) = \operatorname{cis}(2\pi) = 1$; one sees that ζ is a primitive n^{th} root of unity. Thus primitive roots of unity exist for every n. As m ranges from 0 to $n-1$, we obtain distinct complex numbers ζ^m, all of which are n^{th} roots of unity. These are all of the n^{th} roots of unity; thus for each $n \in \mathbb{N}$, there are *exactly* n distinct n^{th} roots of unity.

If one graphs the n^{th} roots of unity in the complex plane, the points lie on the unit circle and they are the vertices of a regular n-gon, with one vertex always at the point $1 = 1 + i0$.

Let $z = r\operatorname{cis}(\theta)$. Then z has exactly n distinct n^{th} roots; they are

$$\sqrt[n]{z} = \zeta^m \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right), \qquad \text{where} \qquad \zeta = \operatorname{cis}\left(\frac{2\pi}{n}\right) \text{ and } m \in \{0, \dots, n-1\}.$$

The algebraic importance of the complex numbers, and the original motivation for their study, is exemplified by the next theorem. This was first conjectured in the 1500's, but was not proven until the doctoral dissertation of Carl Friedrich Gauss in 1799 at the age of 22. Incidentally, was the first to prove the constructibility of a regular 17-gon, at an even earlier age.

Theorem 2. (The Fundamental Theorem of Algebra)
Every polynomial with complex coefficients has a zero in $\mathbb{C}$.

From this, it follows that every polynomial with complex coefficients factors completely into the product of linear polynomials with complex coefficients.

5. Exercises

The *rectangular form* of a complex number is $z = a + bi$.
The *polar form* of a complex number is $z = r\operatorname{cis}\theta$.

Exercise 1. Let $z = 7 - 2i$ and $w = 5 + 3i$.
Compute the following, expressed in rectangular form.

- **(a)** $z + w$
- **(b)** $3z - 8w$
- **(c)** zw
- **(d)** $\frac{z}{w}$
- **(e)** $\overline{z}$ and $|z|$

Exercise 2. Find the rectangular and polar forms of all sixth roots of unity.

Exercise 3. Find the rectangular and polar forms of all solutions to the equation $z^6 - 8 = 0$.

Exercise 4. Find the rectangular and polar forms of all solutions to the equation $z^6 - a = 0$, where $a = \sqrt{3} + i$.

Exercise 5. Find all complex solutions to the equation $z^9 - 1 = 0$.

Exercise 6. Find all complex solutions to the equation $z^5 = i$.

Exercise 7. Find all complex numbers z such that $z^5 = 32$.

Exercise 8. Find all complex numbers z such that $z^3 = 1 + i$

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